# THE DIRECT METHOD OF LIAPUNOV IN STABILITY PROBLEMS OF ELASTIC SYSTEMS

## (O PRIAMOM METODE LIAPUNOVA V ZADACHAKH USTOICHIVOSTI UPRUGIKH SISTEM)

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Considered is the problem of stability of the plane state of an elastic thin plate of infinite length simply supported along two edges (beam), subjected to the action of constant forces in its plane, from the point of view of application of various methods of analysis, namely the methods of direct integration and the direct method of Liapunov. The definition of stability in the sense of Liapunov is given for the problem under discussion, and the theorems of the direct method of Liapunov regarding stability and instability [1, 2] are given; to this end an auxiliary metric space is introduced, in order to construct in it the corresponding functionals (see the dissertation of Krasovskii, and also the paper [3]).

It is assumed, that the equations of motion for the dimensionless deflection $w(x, t)$, referred to the chord $a$ of the plate, may be written in the form

$$\frac{\partial^4 w}{\partial x^4} - \frac{a^2 N}{D}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad w(x, t) = \frac{\partial^2 w(x, t)}{\partial x^2} = 0 \quad \text{for } x = 0,1 \tag{1}$$

Here $x$ is the dimensionless space coordinate, referred to the chord $(0 < x < 1)$, $t$ is a dimensionless time, referred to the quantity $(\mu a^4/D)^{1/2}$ $\mu$ is the mass per unit of area, $D$ is the rigidity, $N$ is the force in the plane of the plate, positive in case of extension.

1. In a static investigation, the plane state of elastic equilibrium of the plate $w(x \equiv 0$ is said to be stable, if there is no other state of elastic equilibrium $w(x) \not\equiv 0$, infinitely close to it ([4], p. 94). Nontrivial solutions $w_m(x) = c_m \sin m\pi x$ $(m = 1, 2, \ldots)$ of equations (1), close to the trivial one for small values of arbitrary constants $c_m$, exist only if the conditions $N = N_m (m = 1, 2, \ldots)$ are satisfied, where $N_m = -(m^2\pi^2 D/a^2)$ is the critical Euler force of order $m$.

From this, using the definition given above, instability should be concluded for $N = N_m (m = 1, 2, \ldots)$ and stability, if the condition $N > N_1$ is satisfied, or any of the conditions $N_{m+1} < N < N_m$. The latter contradicts the well-known experimental fact of plate buckling for $N < N_1$.

**2.** In a dynamic investigation, the plane undisturbed state of the plate $w(x, t) \equiv 0$ is said to be stable, if among the solutions of equations (1) of the type

$$w(x, t) = X(x) e^{\omega t}, \qquad w(x, t) = [X_1(x) + tX(x)] e^{\omega t}, \ldots$$

or solutions, obtained from the preceding ones by separation of real and imaginary parts (natural and forced motions) there are no divergent ones (with an amplitude increasing to infinity). All solutions of this type are easily found:

$$w_m(x, t) = c_m \sin m\pi x \cos q_m t, \qquad q_m = m^2\pi^2 \left(1 + \frac{a^2 N}{m^2\pi^2 D}\right)^{1/2}$$

$$w_m'(x, t) = c_m' \sin m\pi x \, \frac{\sin q_m t}{q_m} \qquad (m = 1, 2, \ldots)$$

($c_m$ are arbitrary constants). Among them there are no divergent ones, if the condition $N > N_1$; if this condition is violated, there are divergent solutions for certain $m$'s of the type

$$w_m(x, t) = c_m \sin m\pi x \operatorname{ch} |q_m| t, \qquad w_m'(x, t) = c_m' \sin m\pi x \, \frac{\operatorname{sh} |q_m| t}{q_m}$$

$$w_m''(x, t) = c_m'' t \sin m\pi x$$

Using the definition of "dynamic" stability, given in Section 2, it is concluded that stability will obtain for $N > N_1$ and instability in the opposite case.

The preceding study is confined to consideration of solutions of a given form. Supplementary information regarding stability may be obtained, by considering sufficiently smooth solutions $w(x, t)$ of arbitrary type, which, together with the derivatives entering into equations (1), may be represented by uniformly converging series of natural and forced motions:

$$w(x, t) = \sum_{m=1}^{\infty} \sin m\pi x \left(c_m \cos q_m t + c_m' \, \frac{\sin q_m t}{q_m}\right)$$

Estimating the coefficient of the series, it may be shown that if $N > N_1$ is satisfied, one can find such $\delta > 0$ for an arbitrary $\epsilon > 0$ and depending only on $\epsilon$, that any sufficiently smooth solution $w(x, t)$, satisfying at the initial instant $t_0$ the condition

$$\rho_2(0, w(x, t_0)) = \sup_x \left| \frac{\partial^4 w(x, t_0)}{\partial x^4} \right| + \sup_x \left| \frac{\partial^3 w(x, t_0)}{\partial x^2 \partial t} \right| < \delta$$

satisfies the condition, for all $t \geqslant t_0$*

$$\rho_1(0, w(x, t)) = \sup_x |w(x, t)| + \sup_x \left|\frac{\partial w(x, t)}{\partial t}\right| < \varepsilon$$

We emphasize that the dynamic stability of the plate for $N > N_1$ does not indicate the possibility of selecting for a given $\epsilon > 0$ such a $\delta > 0$, depending only on $\epsilon$, that any solution $w(x, t)$, satisfying at the initial instant $t_0$ the condition $\rho_1(0, w(x, t)) < \delta$ will satisfy the condition $\rho_1(0, w(x, t)) < \epsilon$ for all $t \geqslant 0$.

In fact, let us consider all the analytical solutions

$$w_n(x, t) = c \sin n\pi x \cos q_n t \qquad (n = 1, 2, \ldots)$$

where the $c$'s are arbitrary numbers. It is easy to calculate that for a given $\epsilon > 0$ the solution $w_n(x, t)$ satisfies the condition

$$\rho_1(0, w(x, t)) < \epsilon$$

for all $t \geqslant 0$ if, and only if, the condition

$$\rho_1(0, w_n(x, 0)) < \delta_n = \varepsilon / (1 + q_n^2)^{1/2}$$

is satisfied at the initial instant $t = 0$.

Since $\delta_n \to 0$ as $n \to \infty$ and $\rho_1(0, w_n(x, 0)) = |c|$, no matter how small $\delta > 0$ is chosen, any solution $w_n(x, t)$ for $0 < c < \delta$ satisfies the condition $\rho_1(0, w_n(x, 0)) < \delta$, however, the solutions $w_n(x, t)$ with sufficiently large orders $n$, for which $\delta_n < c$ satisfy condition $\rho_1(0, w(x, t)) < \epsilon$ not for all $t \geqslant 0$.

The indicated feature, characteristic of elastic systems, may be made plausible in the following manner: the condition $\rho_1(0, w(x, t_0)) < \delta$, constraining the initial deflections and velocities of points of the plate, does not limit the initial potential strain,** which, in the process of motion, passes over into kinetic energy, and produces, at isolated instants, "splashes" of magnitude $\rho_1(0, w(x, t))$. In order to suppress these splashes, it is sufficient to impose a more rigid constraint $\rho_2(0, w(x, t_0)) < \delta$ on the initial state of the plate, which would limit not only the initial deflections, and velocities, but also the corresponding initial energy of bending strains and strain rates.

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* Compare with the definition of correctness ([5], pp. 80-83).

** It is easy to calculate that the potential strain energy, corresponding to the initial deflection $w_n(x, 0) = c \sin n\pi x$ of the plate increases to infinity as $n$ increases.

**3.** Before we apply the direct method of Liapunov to the problem under consideration, let us discuss one of possible versions of defining and proving basic theorems, which may be more convenient in certain applications, than those contained* in [3].

In the metric [6] space $R(a, \rho)$, whose elements are determined by $a, a_0, a', \ldots$, there is given a continuous curve $a(a_0, t_0, t)$, emanating from the point $a_0$ at the instant of time $t_0$ if there corresponds to each given value of the real parameter (of time) $t$ in the interval $t_0 \leqslant t < \infty$ in $R(a, \rho)$ a determined point $a(a_0, t_0, t)$, such that $a(a_0, t_0, t_0) = a$ and the reflection $a(a_0, t_0, t)$ are continuous** for arbitrary $t \geqslant t_0$. Reduced continuous curves $a(a_0, t_0, t)$, given for a finite interval of time $t_0 \leqslant t \leqslant t$, (each curve having its own interval) are also considered. If at the point $a_0$ at the instant of time $t_0$ more than one curve is emanating, we write $a_\alpha(a_0, t_0, t)$, indicating different curves by different values of the index $\alpha$ and calling the pencil of curves $a_\alpha(a_0, t_0, t)$ the set of all curves, which emanate from the point $a_0$ at the instant of time $t_0$. Different pencils of curves $a_\alpha(a_0, t_0, t)$, emanating from different points $a_0$ at different instants of time $t_0 \geqslant 0$ are considered.

Among the curves considered, we separate a class $L$ of curves which satisfy certain supplementary conditions. These supplementary conditions, which in specific problems may be differential equations with ordinary or partial derivatives, integra-differential equations, boundary conditions, smoothness conditions, etc., are written down in the form $L(a, t) = 0$ and are called, by convention, the equations of the boundary value problem. It is assumed, that there exists a curve of class $L$, to which, for any $t \geqslant 0$ there corresponds a point $a'$ in $R(a, \rho)$.

This curve we call the undisturbed motion $a'$, and the remaining curves of class $L$ are called the disturbed motions. We assume also that for any $\delta > 0$ one can find at least one disturbed motion $a(a_0, t_0, t)$ initiating in the vicinity $\rho(a', a_0) < \delta$.

On any set of pairs $(a, t)$ there is determined a real functional $f(a, t)$, if there corresponds to each pair $(a, t)$ of this set a definite (one for any given pair and finite) real number $f(a,t)$.

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* In [3], as a result of excessive generality, certain proofs (for example, the sufficiency in the stability theorem) are not in a form in which a simple reference to them may fully satisfy the reader.

** This means, that $\rho(a(a_0, t_0, t), a(a_0, t_0, t_n)) \to 0$ as $n \to \infty$ for any sequence $t_n$, $t_n \geqslant t_0$ converging to $t$.

Let the pair $(a, t)$ be such, that $\rho(a', a) < R$, where $R$ is a positive fixed number and let a disturbed motion $a(a, t, t + \tau)$, $\tau \geqslant 0$ exist, which initiates at the point $a$ at the instant of time $t$. The set of all such pairs $(a, t)$ is designated by $LRT$. We consider the functional $f(a, t)$ determined on the set $LRT$ and possessing on $LRT$ certain properties. The functionals $f(a, t)$ may not possess these properties outside of $LRT$, where they may also be determined. Since each pair $(a, t) \in LRT$ corresponds to a point on a certain disturbed motion of the boundary value problem $L(a, t) = 0$, we shall talk occasionally about this or that property of the functionals $f(a, t)$ valid on $LRT$, as a property valid by virtue of the equations of the boundary value problem $L(a, t) = 0$ (or as a property valid along disturbed motions).

The functional $f(a, t)$ is finite and positive by virtue of the equations of the boundary value problem $L(a, t) = 0$, if for an arbitrary positive number $\epsilon < R$ and for any pair $(a, t)$ $LRT$, satisfying the condition $\rho(a', a) > \epsilon$, one can find such a $\mu > 0$, depending only on $\epsilon$ for the condition $f(a, t) \geqslant \mu$ to be satisfied.

The functional $f(a, t)$ admits, by virtue of the equations of the boundary value problem $L(a, t) = 0$, an infinitely small upper bound, if for any $\mu > 0$ one can find such a $\delta > 0$, depending only on $\mu$, that $|f(a, t)| < \mu$ for any pair $(a, t)$ $LRT$ satisfying the condition $\rho(a', a) < \delta$.

The functional $f(a, t)$ is called non-increasing by virtue of the equations of the boundary value problem $L(a, t) = 0$, if on $LRT$ along any disturbed motion $a_{\alpha}(a_0, t_0, t)$ the function $f(a_{\alpha}(a_0, t_0, t)t$ does not increase with the increase of $t$.

The functionsl $f(a, t)$ is called vanishing along the curve $a_{\alpha}(a_0, t_0, t)$, if the function $f(a_{\alpha}(a_0, t_0, t), t)$ exists for all $t \geqslant t_0$ and approaches to zero as $t \to \infty$.

The region $f(a, t) > 0$ is called the set of pairs $(a, t)$ $LRT$ for which $f(a, t) > 0$. The functional $f(a, t)$ is called bounded in the region $f(a, t) > 0$, if on $LRT$ for some $N > 0$ there follows the inequality $f(a, t) < N$ from the inequality $f(a, t) > 0$.

The functional $f(a, t)$ has, by virtue of the equations of the boundary value problem $L(a, t) = 0$, a finite positive derivative $f'(a, t)$ in the region $f(a, t) > 0$, if, for $\mu > 0$ and any disturbed motion $a_{\alpha}(a_0, t_0, t)$ which satisfies on $LRT$ the condition $f(a_{\alpha}(a_0, t_0, t), t) > \mu$, one can find such $\nu > 0$, depending only on $\mu$ and possibly, on the disturbed motion $a_{\alpha}(a_0, t_0, t)$, taken that on $LRT$ the inequality $df(a_{\alpha}(a_0, t_0, t), t)/df \geqslant \nu$ is satisfied.

The set of all points of the metric space $R(a, \rho)$, belonging to all possible curves of the pencil of disturbed motions, initiating at the point $a_0$ at the instant of time $t_0$, shall be designated by $A(a_0, t_0)$. To each pair $(a, t)$ $LRT$ there corresponds its own pencil of disturbed motions (initiating at the point $a$ at the instant of time $t$), and as a consequence its own set $A(a, t)$. The upper face of the distances from points $a'$ to points of the set $A(a, t)$ shall be designated as $\rho^0(a', A(a, t))$. Since $a$ $A(a, t)$, then $\rho^0(a', A(a, t)) \geqslant \rho(a', a)$.

*Definition.* The undisturbed motion $a'$ is called stable, if for any $\epsilon > 0$ one can find such a $\delta > 0$, depending only on $\epsilon$, that any disturbed motion $a_\alpha(a_0, t_0, t)$, initiating in the vicinity of $\rho(a', a_0) < \delta$, satisfies for any $t \geqslant t_0$ in the region of its definition* the condition $\rho(a', a_\alpha(a_0, t_0, t)$. In the opposite case the undisturbed motion $a'$ is called unstable.

It is obvious that if the undisturbed motion $a'$ is stable one can find for any $\epsilon_1 > 0$ such a $\delta_1 > 0$, depending only on $\epsilon_1$, that any pencil of disturbed motions, initiating at the instant $t$ at the point $a$ from the vicinity $\rho(a', a) < \delta_1$, satisfies the condition $\rho^0(a', A(a, t)) < \epsilon_1$.

Passing to the proof of the theorems, we emphasize once more that we use the fact, in our discussion, regarding the existence of disturbed motions in an arbitrarily small vicinity of the undisturbed one. In those specific problems, in which the questions regarding existence are not clarified, the results proved have only a conventional meaning: if the corresponding solutions exist, and the conditions of the theorems are satisfied, then the conclusions are also valid. As a consequence, we apply a scheme in which the questions of existence are separated from the questions of stability in the same manner, as this is frequently done in studying questions of existence and uniqueness.

*Stability Theorem.* In order that the undisturbed motion be stable, it is necessary and sufficient that there exists, by virtue of the equations of the boundary value problem, a finite positive non-increasing functional, which admits an infinitely small upper bound.

*Proof. Necessity.* Let the undisturbed motion $a'$ be stable. We take some $E > 0$ and by virtue of stability we find such $R > 0$, depending only on $E$, that any pencil of disturbed motions, initiating at the instant of time $t$ at the point $a$ from the vicinity $\rho(a', a) < R$ satisfies the condition $\rho^0(a', A(a, t)) < E$. To each pair $(a, t)$ $LRT$ we establish a corresponding definite (one for a given pair and finite) real number $f(a, t) = \rho^0(a', A(a, t))$.

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* Damped disturbed motions are also admitted.

The functional $f(a, t)$ is finite positive, since the inequality $\rho^0(a', A(a, t)) \geqslant \rho(a', a)$ is satisfied.

The functional $f(a, t)$ admits an infinitely small upper bound, since, by virtue of stability, for an arbitrary $\mu > 0$ one can find such a positive $\delta < R$ depending only on $\mu$, that any pencil of disturbed motions, initiating at the instant $t$ at the point $a$ from the vicinity $\rho(a', a) < \delta$ satisfies the condition $\rho^0(a', A(a, t)) < \mu$ that is, from $\rho(a', a) < \delta$ it follows $|f(a, t)| < \mu$.

Assume that for any $t$ in the interval $t_1 < t \leqslant t_2$, $t_1 > t_0$ the points of disturbed motion $a_\alpha(a_0, t_0, t)$, including the terminal ones $a_1 = a_\alpha(a_0, t_0, t_1)$ and $a_2 = a_\alpha(a_0, t_0, t_2)$, satisfy the condition $\rho(a', a_\alpha(a_0, t_0, t)) < R$. Since the pencils, originating at the points of one and the same curve, enter completely, at a later instant of time, into the group of pencils, originating at an earlier instant of time, it follows $A(a_1, t_1) \supseteq A(a_2, t_2)$. From here $\rho^0(a', A(a_1, t_1)) > \rho^0(a', A(a_2, t_2))$, that is, $f(a_1, t_1) > f(a_2, t_2)$ or, more completely, $f(a_\alpha(a_0, t_0, t_1), t_1) \geqslant f(a_\alpha(a_0, t_0, t_2), t_2)$, which indicates that the functional $f(a, t)$ does not increase by virtue of the equations of the boundary value problem $L(a, t) = 0$.

*Supplement regarding asymptotic stability.* Let the undisturbed motion be stable and let, in addition, any undamped disturbed motion, sufficiently close to the undisturbed one, approach it asymptotically. Then the functional $f(a, t) = \rho^0(a', A(a, t))$, constructed on $LRT$, vanishes along any such disturbed motion.

In fact, under the assumptions met, for some positive $\delta < R$ any undamped disturbed motion $a_\alpha(a_0, t_0, t)$, initiating in the vicinity of $\rho(a', a_0) < \delta$, satisfies the conditions $\rho(a', a_\alpha(a_0, t_0, t)) < R$ for any $t \geqslant t_0$ and $\rho(a', a_\alpha(a_0, t_0, t)) \to 0$ for $t \to \infty$. Let us take some undamped disturbed motion $a_\alpha(a_0, t_0, t)$ initiating in the vicinity of $\rho(a', a_0) < \delta$. The first condition guarantees the existence of the function $f(a_\alpha(a_0, t_0, t), t)$ for any $t \geqslant t_0$. The second condition means that for any $\nu_1 > 0$ one can find such $t_1 = t_1(a_0, t_0, a) > t_0$ that for all $t > t_1$ the inequality $\rho(a', a_\alpha(a_0, t_0, t)) < \nu_1$ is satisfied.

Let an arbitrary $\mu_1 > 0$ be given. By virtue of stability for the given $\mu_1$ one can find such a $\nu_1 > 0$ that any pencil of disturbed motions initiating at the instant of time $t$ at the point $a$ from the vicinity $\rho(a', a) < \nu_1$ satisfies the condition $\rho^0(a', A(a, t)) < \mu_1$. For this $\nu_1$ one can find, as is indicated above, such a $t_1 = t_1(a_0, t_0, a) \geqslant t_0$ that for all $t > t_1$ at points $a = a_\alpha(a_0, t_0, t)$ of the curve taken, the inequality $\rho(a', a) < \nu_1$ is satisfied, and consequently, $\rho^0(a', A(a, t)) < \mu$, that is $f(a, t) = f(a_\alpha(a_0, t_0, t), t) < \mu_1$.

Thus, for any undamped disturbed motion $a_a(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$ for a given $\mu_1 > 0$ one can find such a $t_1 = t_1(a_0, t_0, a) > t_0$ that $f(a_a(a_0, t_0, t), t) < \mu_1$ for all $t > t_1$ that is $f(a_a(a_0, t_0, t) \to 0$ for $t \to \infty$.

If $\rho(a', a_a(a_0, t_0, t)) \to 0$ as $t \to \infty$ uniformly with respect to $a$ and $a_0$ from the vicinity $\rho(a', a_0) < \delta$, the number $t_1$ which was mentioned above, may be selected as being independent of $a$ and $a_0$ from the vicinity $\rho(a', a_0) < \delta$.

*Sufficiency.* Assume that for some $R > 0$ the functional $f(a, t)$ possesses on $LRT$ all the properties indicated in the theorem. Let also be given a positive $\epsilon < R$.

The functional $f(a, t)$ is finite positive, therefore for $\epsilon > 0$ and any pair $(a, t)$ $LRT$ satisfying* the condition $\rho(a', a) \geqslant \epsilon$ one can find such a $\mu > 0$, depending only on $\epsilon$, that $f(a, t) > \mu$ is satisfied.

The functional $f(a, t)$ admits an infinitely small upper bound, therefore, the number $\mu > 0$ permits to determine such a positive $\delta < \epsilon$, depending only on $\mu$ so that $|f(a, t)| < \mu$ and that for any pair $(a, t)$ $LRT$ the condition $\rho(a', a_0) < \delta$ is satisfied.

Let us prove that for the $\delta$ found, the disturbed motion $a_a(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$, satisfies for all $t \geqslant t_0$ the inequality $(\rho(a', a_a\, a_0, t_0, t)) < \epsilon$ in the region of its definition. We shall assume that this is not so, and that there exists a disturbed motion $a_a(a_0, t_0, t)$, initiating in the vicinity of $\rho(a', a_0) < \delta$ which, at a certain instant of time $t > t_0$, does not satisfy the condition $\rho(a', a_a(a_0, t_0, t)) < \epsilon$.

Due to continuity of the curve $a_a(a_0, t_0, t)$ one can find such a $t_1 > t_0$ that in the interval $t_0 < t < t_1$ the inequality $\rho(a', a_a(a_0, t_0, t) < \epsilon$ is satisfied, and at the instant $t_1$ the equality $\rho(a', a_a(a_0, t_0, t_1)) = \epsilon$

Then $f(a_a(a_0, t_0, t_1), t_1) \geqslant \mu$, which contradicts the condition of divergence of the function $f(a_a(a_0, t_0, t), t)$ determined for any $t$ in the interval $t_0 < t < t$, and taking on at its lower limit the value $f(a_0, t_0) < \mu$.

*Supplement regarding asymptotic stability.* Assume that the functional $f(a, t)$ for some $R > 0$ possesses on $LRT$ all the properties indicated in the theorem and assume, in addition, that the functional $f(a, t)$ vznishes along any undamped disturbed motion, sufficiently close to the undisturbed one. Then any undamped disturbed motion, sufficiently close to the

* If for a given $\epsilon$ there is no pair $(a, t)$ $LRT$, satisfying condition $\rho(a', a) > \epsilon$, the proof is trivial.

undisturbed one, approaches it asymptotically.

In fact, under the assumptions met, for some positive $\delta < R$ any undamped disturbed motion $a_\alpha(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$, satisfies the conditions $\rho(a', a_\alpha(a_0, t_0, t)) < R$ for all $t \geqslant t_0$ and $f(a_\alpha(a_0, t_0, t), t) \to 0$ as $t \to \infty$.

Let there be given any positive $\epsilon_1 < R$. The functional $f(a, t)$ is finite positive, therefore, for the given $\epsilon_1$ and for any pair $(a, t)$ $LRT$ satisfying* the condition $\rho(a', a) > \epsilon_1$, one can find such a $\mu_1 > 0$ depending only on $\epsilon_1$ that $f(a, t) > \mu_1$.

Let us take some undamped disturbed motion $a_\alpha(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$. Since $f(a_\alpha(a_0, t_0, t), t) \to 0$ as $t \to \infty$, one can find such a $t_1 = t_1(a_0, t_0, a) \geqslant t_0$ that $f(a_\alpha(a_0, t_0, t), t) < \mu_1$ for all $t > t_1$. Then for all $t > t_1$ for points $a = a_\alpha(a_0, t_0, t)$ of the curve taken, the relationship $\rho(a', a) = \rho(a', a_\alpha(a_0, t_0, t)) < \epsilon_1$ is satisfied because, if for some $t > t_1$ the inequality $\rho(a', a) \geqslant \epsilon_1$ would hold, for this $t > t_1$ one would also have $f(a, t) = f(a_\alpha(a_0, t_0, t), t) \geqslant \mu_1$, which is impossible.

Thus, for any undamped disturbed motion $a_\alpha(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$ for a given $\epsilon_1 > 0$, one can find such a $t_1 = t_1(a_0, t_0, a) \geqslant t_0$ that $\rho(a', a_\alpha(a_0, t_0, t)) < \epsilon_1$ for all $t > t_1$ that is $\rho(a', a_\alpha(a_0, t_0, t)) \to 0$ as $t \to \infty$.

If the functional $f(a, t)$ vanishes along undamped disturbed motions $a_\alpha(a_0, t_0, t)$, initiating in the vicinity $\rho(a', a_0) < \delta$, uniformly with respect to $a$ and $a_0$ from the vicinity $\rho(a', a_0) < \delta$ the number $t_1$, which was mentioned above, may be selected as being independent of $a$ and $a_0$, from the vicinity $\rho(a', a_0) < \delta$, that is, all undamped disturbed motions, sufficiently close to the undisturbed one, approach it uniformly asymptotically.

*Instability Theorem.* Let us assume that for some $R > 0$ among the disturbed motions, initiating on $LRT$, there are no damped ones, and that any two portions of disturbed motions connected on $LRT$, having at some instant of time in their region of definition a common point, coincide (on $LRT$) at all the subsequent instants of time, that is, that for such portions $a(a_1, t_1, t)$ and $a(a_2, t_2, t)$ from $a(a_1, t_1, t_3) = a(a_2, t_2, t_3)$ it follows $a(a_1, t_1, t) \equiv a(a_2, t_2, t)$ for $t \geqslant t_3$.

In order that the undisturbed motion be unstable, it is necessary and sufficient, that there exists a functional $f(a, t)$ which is bounded in

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* If for the given $\epsilon_1$ there is not a single pair $(a, t)$ $LRT$ satisfying the condition $\rho(a', a) > \epsilon_1$, then the proof is trivial.

the region $f(a,\ t) > 0$ having pairs $(a,\ t)$ for any sufficiently small vicinity of undisturbed motion, and that it has, by virtue of the equations of the boundary value problem, a finite positive derivative $f'(a, t)$.

*Proof. Necessity.** Let the undisturbed motion $a'$ be unstable. Then for some $R > 0$ and an arbitrary positive $\delta < R$, we can find a disturbed motion $a(a_0,\ t_0,\ t)$, initiating in the vicinity $\rho(a',\ a_0) < \delta$ and satisfying the condition $\rho(a',\ a(a_0,\ t_0,\ t) < R$ within some finite interval $t_0 \leqslant t \leqslant t^0$, whereby $\rho(a',\ a(a_0,\ t_0,\ t^0)) = R$.

The portion of disturbed motion $a(a_0,\ t_0,\ t)$, $t_0 \leqslant t < t^0$ shall be called connected on $LRT$ if for any $t$ of the interval $t_0 \leqslant t < t^0$ the relation $\rho(a',\ a(a_0,\ t_0,\ t) < R$ is satisfied whereby either

$$\rho(a',\ a(a_0,{}_0t\ ,\ t^0)) = R, \text{ or } t^0 = \infty.$$

By virtue of uniqueness, mentioned in the condition of the theorem, the set of all portions of disturbed motions connected on $LRT$ may be subdivided into classes $l_\beta$ (a given index $\beta$ characterizes a given class), including in a class with some portion $a(a_0,\ t_0,\ t)$ all the portions, having with it, at any instant of time $t \geqslant t_0$, common points ([6], p. 17). For some portion $a(a_0,\ t_0,\ t)$ of class $l_\beta$ the region of its definition is the interval $t_0 < t < t_\beta{}^0$, whereby $t_\beta{}^0$ has a common value for all portions of the class $l_\beta$ (for example, the portion $a(a_1,\ t_1,\ t)$ of class $l_\beta$ is defined in the interval $t_1 \leqslant t < t_\beta{}^0$).

If $t_\beta{}^0 < \infty$ along any portion of class $l_\beta$ in the region of its definition, we put $f(a,\ t) = \exp(t - t_\beta{}^0)$. For example, along the portion $a(a_0,\ t_0,\ t)$ of class $l_\beta$ we put $f(a(a_0,\ t_0,\ t),\ t) = \exp(t - t_\beta{}^0)$ for any $t$ within the interval $t_0 < t < t_\beta{}^0$. If $t_\beta{}^0 = \infty$ along any portion of class $l_\beta$ in the region of its definition, we put $f(a,\ t) \equiv 0$.

Thus, to each pair $(a,\ t)$ $LRT$ there corresponds a well-determined (one for a given pair and finite) real number $f(a,\ t)$.

Pairs $(a,\ t)$ $LRT$ for which $f(a,\ t) = \exp(t - t_\beta{}^0) > 0$ exist for any sufficiently small neighborhood of undisturbed motions (see beginning of proof).

$0 \leqslant f(a,\ t) < 1$ is satisfied everywhere on $LRT$, that is, the functional $f(a,\ t)$ is bounded.

The functional $f(a,\ t)$ is determined on $LRT$ in such a manner that along any disturbed motion $a(a_0,\ t_0,\ t)$ the relation

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* The necessary conditions can be proved without the assumption that the disturbed motions are undamped.

$$df(a(a_0, t_0, t), t)/dt = f(a(a_0, t_0, t), t)$$

is satisfied. It follows that in the region $f(a, t) > 0$ the functional possesses, by virtue of the equations of the boundary value problem, a finite positive derivative $f'(a, t)$.

*Sufficiency*. Let for some $R > 0$ the functional $f(a, t)$ exist on $LRT$, which possesses the properties indicated in the conditions of the theorem. We assume that the undisturbed motion $a'$ is stable. Then one can find such a $\delta$ that any disturbed motion $a(a_0, t_0, t)$ initiating in the vicinity of $\rho(a', a_0) < \delta$ satisfies for all $t \geqslant t_0$ the condition $\rho(a', a(a_0, t_0, t)) < R$ and does not pass beyond the region in which the functional $f(a, t)$ possesses the properties indicated in the conditions of the theorem.

According to the conditions of the theorem, one can select such an undamped motion $a(a_0, t_0, t)$ among the disturbed motions initiating in the vicinity of $\rho(a', a_0) < \delta$, that $f(a_0, t_0) \geqslant \mu > 0$. We conclude, from the fact that the functional $f(a, t)$ possesses in the region $f(a, t) > 0$, by virtue of the equations of the boundary value problem, a finite positive derivative $f'(a, t)$, that along the selected disturbed motion $a(a_0, t_0, t)$ for $t \geqslant t_0$ the relations

$$f(a(a_0, t_0, t), t) \geqslant \mu, \qquad df(a(a_0, t_0, t), t)/dt \geqslant \nu$$

are satisfied simultaneously, where $\nu$ is some positive number. Then it follows from the boundedness of the functional $f(a, t)$ in the region $f(a, t) > 0$, and from the first inequality, that for some $N > 0$ and any $t \geqslant t_0$ the relation $f(a(a_0, t_0, t), t) < N$ is fulfilled, while from the second inequality the contradicting relation $f(a(a_0, t_0, t), t) \geqslant f(a_0, t_0) + \nu(t - t_0)$ follows for sufficiently large $t > t_0$.

*Remark*. The uniqueness mentioned in the condition of the theorem was not used in proving sufficiency. From the conditions of existence merely the fact that undamped disturbed motions $a(a_0, t_0, t)$ exist, which is initiated in a sufficiently small neighborhood of the undisturbed motion in the region $f(a, t) > 0$, was employed.

**4.** We consider the set $W$ of real functions $w(x, t)$, completely defined and continuous in $x$, $t$ in the region $0 < x < 1$, $t > t_0 > 0$, together with the derivatives $w_{xx}(x, t)$, $w_x(s, t)$, $w_1(s, t)$. At a fixed instant of time $t$ we establish a correspondence between the function $w(x, t)$ $W$ in the region of its definition and the point $w = [w(x, t), w_t(s, t)]$, a pair of functions of magnitude $x$. The points $w_1 = [w_1(x, t_1), w_{1t1}(x, t_1)]$ and $w_2 = [w_2(x, t_2), w_{2t2}(x, t_2)]$ shall be called coincident, if $w_1(s, t_1) \equiv w_2(x, t_2)$ and $w_{1t1}(x, t_1) \equiv w_{2t2}(x, t_2)$. The distance between the points $w_1$ and $w_2$ is called the non-negative number

$$\rho(w_1, w_2) = \left\{\int_0^1 dx \sum_{k=0}^{2}\left[\frac{\partial^k w_1(x,t_1)}{\partial x^k} - \frac{\partial^k w_2(x,t_2)}{\partial x^k}\right]^2 + \int_0^1 dx\left[\frac{\partial w_1(x,t_1)}{\partial t_1} - \frac{\partial w_2(x,t_2)}{\partial t_2}\right]^2\right\}^{1/2}$$

It is easily verified that $\rho(w_1, w_2)$ satisfies all the axioms of metric space. The set of points $w$, together with the metric $\rho(w_1, w_2)$, forms the metric space $R(w, \rho)$. The set of points $w$ corresponding to functions $w(x, t)$ $W$ for all possible $t$ within the region of definition, form in $R(w, \rho)$ a continuous curve which we identify in the following with the function $w(x, t)$ itself.

We shall call the conditions of the boundary value problem $L(w, t) = 0$ the equations (1) and the supplementary conditions of smoothness, that is, the continuity for all $x$, $t$ of the functions $w_{xxxx}(x, t)$, $w(x, t)$, $w_{xxt}(x, t)$, $w_{xt}(x, t)$, $w_{tt}(x, t)$. From the set $W$ we isolate a class $L$ of curves $w(x, t)$ which satisfy the conditions of the boundary value problem $L(w, t) = 0$. The curve $w(x, t) \equiv 0$ of the class $L$ to which there corresponds for any $t \geqslant 0$ in $R(w, \rho)$ a fixed point $O$, we shall call the undisturbed motion; the remaining curves of class $L$ shall be called disturbed motions.

In accordance with the definition in Section 3, the undisturbed motion $w(x, t) \equiv 0$ shall be called stable, if for any $\epsilon > 0$, we can find such a $\delta > 0$, depending only on $\epsilon$, that any disturbed motion $w(x, t)$ initiating in the vicinity of $\rho(O, w(x, t_0)) < \delta$ satisfies for $t \geqslant t_0$ the condition $\rho(O, w(x, t)) < \epsilon$. To study stability we consider the functional

$$f(w) = \int_0^1 dx\left(w_{xx}^2 + \frac{a^2N}{D} w_x^2 + w_t^2\right)$$

It can be shown that for any function $w(x, t)$ $W$, satisfying the boundary conditions and the conditions of smoothness of the boundary value problem $L(w, t) = 0$, the following inequalities are satisfied

$$\int_0^1 dx\, w_{xx}^2 \geqslant \pi^2 \int_0^1 dx\, w_x^2, \qquad \int_0^1 dx\, w_x^2 \geqslant \pi^2 \int_0^1 dx\, w^2$$

with the aid of which we obtain

$$f(w) \geqslant \gamma\rho^2(0, w), \qquad \gamma = \tfrac{1}{4}\min\left\{1; \left(1 + \frac{a^2N}{\pi^2 D}\right);\ 2\pi^2\left(1 + \frac{a^2N}{\pi^2 D}\right)\right\}$$

From this we conclude, that if the condition $1 + (a^2N/\pi^2 D) > 0$ is satisfied, that is, if $N > N_1$ the functional $f(w)$ is finite positive* by virtue of the boundary value problem $L(w, t) = 0$. The functional $f(w)$ admits an infinitely small upper bound, as is evident from the estimate

$$|f(w)| \leqslant \left(1 + \left|\frac{a^2N}{\pi^2 D}\right|\right)\rho^2(0, w)$$

Along any curve $w(x, t) \in W$ which satisfies the smoothness conditions of the boundary value problem $L(w, t) = 0$ the function $f(w(x, t))$ of time $t$ is determined, whose derivative with respect to $t$ may be written in the form, by means of differentiation under the integral sign and partial integration

$$\frac{d}{dt} f(w(x,t)) = 2\int_0^1 dx \left[\frac{\partial^4 w}{\partial x^4} - \frac{a^2N}{D}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2}\right]\frac{\partial w}{\partial t} +$$
$$+ 2\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\, \partial t} - \frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial t} + \frac{a^2N}{D}\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\right]_0^1$$

It is evident that by virtue of the equations of the boundary value problem $L(w, t) = 0$ the relation $df(w(x, t))/dt \equiv 0$ is fulfilled and, consequently, the functional $f(w)$ does not increase with $t$ along any disturbed motion $w(x, t)$.

In satisfying condition $N > N_1$ stability, in the sense of the definition given above, follows from the first theorem of the direct method of Liapunov.

We note that along disturbed motions

$$|w(x,t)| = \left|\int_0^x dx\, w_x\right| \leqslant \left(\int_0^1 dx\, w_x^2\right)^{1/2} \leqslant \rho(0, w(x,t))$$

Therefore, for $N > N_1$ for given $\epsilon > 0$, one can find such a $\delta > 0$ depending only on $\epsilon$ that any disturbed motion $w(x, t)$ initiating in the vicinity $\rho(O, w(x, t_0)) < \delta$ satisfies for all $t \geqslant t_0$ the condition $\rho_3(O, w(x, t)) = \sup_x |w(x, t)| < \epsilon$.

Let us now consider the functional

$$\varphi(w) = \begin{cases} -f(w)\psi(w) & \text{for } f(w) < 0, \\ 0 & \text{for } f(w) \geqslant 0, \end{cases} \quad \psi(w) = \int_0^1 dx\, ww_t$$

Let us assume that the relation $\psi(w(x, t_0)) \geqslant \mu$ is satisfied for some

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* We emphasize that in proving the finite positive state of the functional $f(w)$ the conditions of smoothness of the curves $w(x, t)$ and the boundary conditions of the boundary value problem $L(w, t)$ h 0 were essentially used.

$\mu > 0$ and for some undamped disturbed motion $w(x, t)$ at instant $t_0$. Then $f(w(x, t_0)) < 0$. Along the disturbed motion considered, this inequality is valid for any $t > t_0$, since $f(w(x, t)) = f(w(x, t_0))$; as a consequence $\phi(w(x, t)) = -f(w(x, t))\psi(w(x, t))$ for any $t > t_0$.

From this we find, by differentiating with respect to $t$ under the integral sign and integrating by parts, using the equations of the boundary value problem $L(w, t) = 0$, that along the disturbed motion considered, for $t > t_0$

$$\frac{d}{dt}\varphi(w(x,t)) \geqslant \nu = f^2(w(x,t_0)) > 0, \qquad \varphi(w(x,t)) \geqslant \mu$$

that is, the functional $\phi(w)$ possesses, by virtue of the equations of the boundary value problem, a finite positive derivative $\phi'(w)$ in the region $\phi(w) > 0$.

The functional $\phi(w)$ is bounded in the region $\phi(w) > 0$ since

$$|\varphi(w)| = |f(w)||\psi(w)| \leqslant \left(1 + \left|\frac{a^2N}{\pi^2 D}\right|\right)\rho^4(0, w)$$

Applying now the instability theorem, we conclude that the undisturbed motion will be unstable, if there exist, in any sufficiently small neighborhood of this motion, undamped disturbed motions initiating at points $w(x, t_0)$ for which $\phi(w(x, t_0)) > 0$.

Let us consider the points $w(x, t_0)$ of the metric space $R(w, \rho)$ which are characterized at the instant of time $t_0$ by deflections $c_0 \sin \pi x$ and velocities $c_1 \sin \pi x$ where $c_0$, $c_1$ are real arbitrary constants (for undamped disturbed motions $w(x, t)$ with initial deflections $w_0(x)$ and velocities $w_1(x)$ to exist, it is sufficient that $w_i(x)$ $(i = 0, 1)$ possess continuous sixth derivatives and that they vanish for $x = 0, 1$, together with the derivatives $w_i''(x)$, $w_i^{IV}(x)$). For these points $w(x, t_0)$, which are found (for suitable $c_0$, $c_1$) in any sufficiently small vicinity of the undisturbed motion, the functional $f(w)$ takes on the values

$$f(w(x,t_0)) = \frac{1}{2}\left[c_0^2\pi^4\left(1 + \frac{a^2N}{\pi^2 D}\right) + c_1^2\right]$$

It is possible to choose such a small $c_1^2$ that for $1 + (a^2N/\pi^2D) < 0$, that is, for $N < N_1$ the relation $f(w(x, t_0)) < 0$ is satisfied; if $c_0c_1 > 0$, the relations $\psi(w(x, t_0)) > 0$ and $\phi(w(x, t_0)) > 0$ are also satisfied. Consequently, for $N < N_1$ the undisturbed motion is unstable.

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